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Published in:
Physical Review Letters

DOI:
10.1103/PhysRevLett.119.030402

Publication date:
2017

Citation for published version (APA):
Burgarth, D., & Ajoy, A. (2017). Evolution-free Hamiltonian parameter estimation through Zeeman markers. *Physical Review Letters*, *119*(3), [030402]. https://doi.org/10.1103/PhysRevLett.119.030402

Evolution-free Hamiltonian parameter estimation through Zeeman markers - Supplementary Material

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(Dated: June 21, 2017)

The essential equation of this article is Eq. (2) which we derive here. Its derivation is elementary but cumbersome and follows standard arguments for Green's functions of rank one perturbations [1, Chapter 6] up to Eq. (7). The remainder of the derivation is analogous to one of Gladwell's inverse problems in vibration [2, Section 4.5]. For completeness we provide a full derivation.

Consider an eigenvalue e' and eigenvector of $|e'\rangle$ of H'. We have the eigenequation

$$H|e'\rangle + f|1\rangle\langle 1|e'\rangle = e'|e'\rangle. \tag{1}$$

Express $|e'\rangle$ in the eigenbasis $\{|e_n\rangle\}$ of H with eigenvalues e_n as

$$|e'\rangle = \sum_{n=1}^{N} \alpha_n |e_n\rangle, \tag{2}$$

which yields

$$f|1\rangle\langle 1|e'\rangle = \sum_{n=1}^{N} (e' - e_n)\alpha_n |e_n\rangle. \tag{3}$$

Upon multiplication with $\langle e_m|$ we obtain

$$\alpha_m = \frac{f\langle e_m|1\rangle\langle 1|e'\rangle}{(e' - e_m)} \tag{4}$$

where we assumed that the spectrum of H and H' have no overlap (this is true for almost all values of f). From Eq. (2) upon multiplication with $\langle 1|$ we arrive at

$$\langle 1|e'\rangle = \sum_{n=1}^{N} \frac{f\langle e_n|1\rangle\langle 1|e'\rangle}{(e' - e_n)} \langle 1|e_n\rangle. \tag{5}$$

Since $\langle 1|e'\rangle \neq 0$ [3], this is equivalent to

$$0 = 1 - \sum_{n=1}^{N} \frac{f|\langle e_n|1\rangle|^2}{(e' - e_n)} \tag{6}$$

or through expansion with $\prod_{m=1}^{N}(e' - e_m)$

$$0 = \frac{\prod_m (e' - e_m) - \sum_n f|\langle e_n|1\rangle|^2 \prod_{m\neq n}(e' - e_m)}{\prod_m (e' - e_m)}. \tag{7}$$

This holds for any eigenvalue e' of H'. The polynomial in x given by the numerator

$$P(x) = \prod_{m=1}^{N} (x - e_m) - \sum_{n=1}^{N} f|\langle e_n|1\rangle|^2 \prod_{m\neq n} (x - e_m) \tag{8}$$

and leading term x^N must therefore factorise as

$$P(x) = \prod_n (x - e'_n), \tag{9}$$

where e'_n are the eigenvalues of H'. We can thus write

$$\begin{aligned} 1 - \sum_{n=1}^{N} & \frac{f|\langle e_n|1\rangle|^2}{(x - e_n)} \\ &= \frac{\prod_m (x - e_m) - \sum_n f|\langle e_n|1\rangle|^2 \prod_{m\neq n}(x - e_m)}{\prod_m (x - e_m)} \\ &= \frac{\prod_n (x - e'_n)}{\prod_m (x - e_m)}. \end{aligned} \tag{10}$$

Multiply with $(x - e_k)$ to obtain

$$(x - e_k) - \sum_{n=1}^{N} \frac{f|\langle e_n|1\rangle|^2 (x - e_k)}{(x - e_n)} = \frac{(x - e_k)\prod_n (x - e'_n)}{\prod_m (x - e_m)} \tag{11}$$

and perform the limit $x \to e_k$ such that

$$-f|\langle e_k|1\rangle|^2 = \frac{\prod_n (e_k - e'_n)}{\prod_{m\neq k}(e_k - e_m)} = (e_k - e'_k) \prod_{m\neq k} \frac{(e_k - e'_m)}{(e_k - e_m)}. \tag{12}$$

Finally we arrive at

$$|\langle e_k|1\rangle|^2 = (e'_k - e_k)/f \prod_{m\neq k} \frac{(e_k - e'_m)}{(e_k - e_m)} \tag{13}$$

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